Abstract: We study $\mu_p$-actions on K3 surfaces in char $p$.

**Preliminaries**

- A K3 surface is a proper smooth (algebraic) surface $X$ over a field with $\Omega^1_X \cong \mathcal{O}_X$ and $H^0(X, \mathcal{O}_X) = 0$.
- An RDP K3 surface is a proper surface $X$ with only RDP (rational double point) singularities whose resolution $\tilde{X}$ is a K3.

**Definition.** An automorphism of a K3 $X$ is **symplectic** if it acts on (the 1dim vector space) $H^0(X, \Omega^1_X)$ trivially.

**Main Def: symplecticness**

Action $\mu_p \smallsetminus \text{Spec } B$ (affine scheme) $\xrightarrow{}$ a decomposition $B = \bigoplus_{z \in \mathbb{Z} / n \mathbb{Z}} B_z$ of vector spaces satisfying $B_i B_j \subset B_{i+j}$.$\xrightarrow{}$ a decomposed $B_{i+j} \subset \bigoplus_{z \in \mathbb{Z} / n \mathbb{Z}} B_{i+j}^z$. We say that $wt(b) = i$ if $b \in B_i$.

**Remark.** If $p \nmid n$, then $\mu_p$-action is equivalent to the action of the cyclic group $\mu(n)$, and $B_i$ is the eigenspace for the $\mu(n)$-action with eigenvalue $i : \mu(n) \rightarrow \mathbb{G}_m$.

**Definition.** We call an action $\mu_p \smallsetminus \tilde{X}$ on an RDP K3 to be **symplectic** if the decomposition of (1dim vector space) $H^0(X, \Omega^1_X)$ is concentrated on $i = 0 \in \mathbb{Z} / n \mathbb{Z}$.

**Remark.** Equivalent to the classical definition if $p \nmid n$ (in which case $\mu(n) \cong \mathbb{Z} / n \mathbb{Z}$).

**Theorem A**

$X$: RDP K3, with an action $\mu_p \smallsetminus X$. symplectic $\implies X/\mu_p \smallsetminus$ RDP K3.

Non-symplectic $\implies X/\mu_p \smallsetminus$ RDP Enriques or rational.

**Proof of A (symplectic case)**

Suffices to consider (symplectic) $\mu_p$-actions. Write $r : X \rightarrow Y = X/\mu_p$. Then $\mathcal{O}_Y = (\mathcal{O}_X)_0$.

**Lemma.** Suppose $z \in X$: a fixed point (smooth or RDP), and $\mu_p$-action symplectic at $z$. Then $z$ is an isolated fixed point, and $\pi(z)$ is an RDP.

**Proof.** If $z$ is smooth, then $\exists x, y$: coordinate with weight $a,b \in \mathbb{Z} / p \mathbb{Z}$, $\neq 0$. We have $a + b = 0$ since symplectic. Then $\Omega(x, y) = k[x^p, y^p]$; RDP of type $A_{p-1}$. If $z$ is an RDP, consider $\mu_p \smallsetminus \mathcal{O}_X \rightarrow$ reduce to the smooth case.

**Cf:** action of finite subgroup of $SL_2(\mathbb{C})$.

**Lemma.** Suppose $z \smallsetminus X$: a non-fixed RDP. Then $\pi(z)$ is smooth or RDP.

**Remark.** This is a new feature.

**Example.** $A_{p-1} : (xy + z^p = 0)$, with action $wt(x, y, 1 + z) = (0, 0, 1)$: quotient is $A_{p-1}$ ($xy + z^p = 0$ where $Z = z^p$).

**Lemma.** Outside the fixed locus and the RDPs, $(\pi(\Omega_2)) = \Omega_2$.

**Proof of C**

If $p \nmid n$, this follows from Keum’s result. We consider the case $p | n$. Write $n = p^e r \cdot \pi r. X$ has an RDP $z$ (since a smooth K3 has no $\mu_p$-action). We may assume it is not fixed by $\mu_{p^e}$ (otherwise, blow it up).

We classify all non-fixed actions on RDPs. And check that $n \in S_0(0) \setminus E_p$ in all cases.

typical case: $p \neq 2$, the $\mu_p$-orbit of $z$ has $r/2$ elements, each of type $A_{p-1}$, and $\mu_2$ acts on $z$ non-symplectically. In this case we deduce $r/2 < b_2(K3) = 22$. This almost implies $n \in S_0(0) \setminus E_{p^e}$, and we kill few exceptions individually.

**Examples for C: $n = 34$**

$X(p): y^2 = x^3 + t^2 x + t^3$ in char $p$.

- $X(0)$ is Kondo’s example of an RDP K3 ($A_1, A_2$) with a $\mu_{34}$-action ($wt(x, y, t) = (4, 23, 6)$) or equivalently a $Z / 34Z$-action.

- $X(17)$ is an RDP K3 ($A_1, A_2, A_{16}$) with a $\mu_{34}$-action (but not with a $Z / 34Z$-action).

- $X(2)$ is not an RDP K3 (a non-RDP singularity the at origin). The true char 2 reduction of $X(0)$ is $y(g' - t^2) = x^3 + t^2 z'$. and does not admit a $\mu_{34}$-action (but instead admits a $Z / 34Z$-action).

**References**


