A NOTE ON MY PAPER ON K3 SURFACES WITH SHIODA–INOSE STRUCTURES

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Abstract. We fix a gap, pointed out by Tianchen Zhao, on my paper [2] on good reduction of some K3 surfaces with Shioda–Inose structures.

0.1. **The involution on product Kummer surfaces over a field.** Let F be a field of characteristic $\neq 2, 3$ and C_1, C_2 be two elliptic curves over F. Suppose that 2-torsion points of C_1 and C_2 are all F-rational. Let $X = \mathrm{Km}(C_1 \times C_2)$ be the Kummer surface attached to $C_1 \times C_2$, that is, the minimal resolution of the quotient $(C_1 \times C_2)/\{[\pm 1]\}$, where $[-1]\colon C_1 \times C_2 \to C_1 \times C_2$ is the inversion map. The minimal resolution is given by the blowup of $(C_1 \times C_2)/\{[\pm 1]\}$ at the images of the 16 2-torsion points of $C_1 \times C_2$. We consider the following divisors of X:

- u_i $(i = 0, 1, 2, 3)$ and v_j $(j = 0, 1, 2, 3)$ are the strict transforms of the images of $\{p_i\} \times C_2$ and $C_1 \times \{q_j\}$, where $\{p_i\} \subset C_1$ and $\{q_j\} \subset C_2$ are the 2-torsion points.
- w_{ij} $(i, j \in \{0, 1, 2, 3\})$ are the exceptional curves above the images of (p_i, q_j).
- $D_0 = v_0 + v_1 + v_2 + 2w_{30} + 2w_{31} + 2w_{32} + 3u_3$.
- $D_\infty = u_0 + u_1 + u_2 + 2w_{03} + 2w_{13} + 2w_{23} + 3v_3$.

Let $\Phi\colon X \to \mathbb{P}^1$ be the elliptic fibration having D_0 and D_∞ as fibers: such a fibration is unique up to automorphisms on $\mathbb{P}^1$. We declare w_{00} to be the zero section of the fibration. Let $\iota_1\colon X \to X$ be the involution that preserves the fibration Φ and induces $[-1]$ on each smooth fiber (in other words, it is the unique extension of the inversion $[-1]$ on the generic fiber). Let $\iota_2\colon X \to X$ be the involution induced by $[1] \times [-1]$ on $C_1 \times C_2$ (or equivalently by $[-1] \times [1]$). Finally, let $\iota = \iota_1 \iota_2\colon X \to X$.

0.2. **Relative version over DVRs.** Let $\mathcal{O}_K$ be a DVR with fraction field K and residue field k, neither of characteristic $2, 3$. For a scheme $\mathcal{X}$ over $\operatorname{Spec} \mathcal{O}_K$, its generic fiber and the special fiber are denoted by $\mathcal{X}_K$ and $\mathcal{X}_k$ respectively.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two elliptic curves over $\mathcal{O}_K$. Suppose that 2-torsion points of $(\mathcal{C}_1)_K$ and $(\mathcal{C}_2)_K$ are all K-rational. Let $\mathcal{Z} \subset (\mathcal{C}_1 \times_{\mathcal{O}_K} \mathcal{C}_2)/\{[\pm 1]\}$ be the closure of the set of the images of the 16 2-torsion points of $(\mathcal{C}_1)_K \times (\mathcal{C}_2)_K$. ($\mathcal{Z}$ consists of 16 components each isomorphic to $\mathbb{P}^1_{\mathcal{O}_K}$.) Then we define $\mathcal{X} = \mathrm{Km}(\mathcal{C}_1 \times_{\mathcal{O}_K} \mathcal{C}_2)$ to be the blowup of $(\mathcal{C}_1 \times_{\mathcal{O}_K} \mathcal{C}_2)/\{[\pm 1]\}$ at the closed subscheme $\mathcal{Z}$.

Define $\mathcal{U}_i, \mathcal{V}_j, \mathcal{W}_{ij}, \mathcal{D}_0, \mathcal{D}_\infty \subset \mathcal{X}$ as in Section 0.1. Then it follows [2, Proposition 2.5] that there is a morphism $\Phi\colon \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$, defined by the "linear system" $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(\mathcal{D}_0)) \cong \mathcal{O}_K^{\oplus 2}$, that the induced morphisms $\Phi_K\colon \mathcal{X}_K \to \mathbb{P}^1_K$ and $\Phi_k\colon \mathcal{X}_k \to \mathbb{P}^1_k$ are as in Section 0.1.

Tianchen Zhao asked me the following.

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Question. Are there involutions ι_1, ι_2, ι on $\mathcal{X}$ that induce the ones as in Section 0.1 on each fiber?

In [2] we stated that we define $\iota \colon \mathcal{X} \to \mathcal{X}$ "similarly", but we realized that this cannot be done in a completely similar way. In this note we fix this gap.

Proposition 0.1. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be as above. Let $X = \mathrm{Km}((\mathcal{C}_1)_K \times (\mathcal{C}_2)_K)$ and let $\iota \in \mathrm{Aut}(X)$ as in Section 0.1. Then, after replacing K with an unramified extension of a bounded degree, there exists a smooth proper ι-model $\mathcal{X}^+$ of X, that is, a smooth proper model $\mathcal{X}^+$ together with an involution $\iota_{\mathcal{X}^+} \colon \mathcal{X}^+ \to \mathcal{X}^+$ that induces ι on $(\mathcal{X}^+)_K = X$.*

We prove this in Section 0.4 after introducing the terminology in Section 0.3.

Here, $\mathcal{X}^+$ is not assumed to be isomorphic to $\mathcal{X}$ given above, but this result is enough to derive the main theorem of [2].

0.3. **Preliminary.** We fix some terminology. Let $\mathcal{O}_K, K, k$ be as in Section 0.2. Let X be a K3 surface over K.

Definition 0.2. A *smooth proper model* of X is a scheme $\mathcal{X}$ smooth and proper over $\mathcal{O}_K$ endowed with an isomorphism $\alpha \colon \mathcal{X}_K \xrightarrow{\sim} X$. Usually the isomorphism α is omitted from the notation.

Suppose $g \in \mathrm{Aut}(X)$ is an automorphism. A *smooth proper g-model* of X is a smooth proper model $(\mathcal{X}, \alpha)$ endowed with an automorphism $g_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X}$ compatible with α (that is, $\alpha \circ (g_{\mathcal{X}})_K = g \circ \alpha$).

Definition 0.3. Now suppose $\mathcal{X}$ is a proper $\mathcal{O}_K$-scheme endowed with a birational equivalence $\alpha \colon \mathcal{X}_K \overset{\sim}{\dashrightarrow} X$, which is not necessarily a morphism. A *g-model* is such an $\mathcal{X}$ endowed with an automorphism (not only a birational self-map) $g_{\mathcal{X}} \colon \mathcal{X} \to \mathcal{X}$ compatible with α, that is, $\alpha \circ (g_{\mathcal{X}})_K = g \circ \alpha$ as rational maps.

Remark 0.4. Although it is often useful to consider smooth proper models that are algebraic spaces (not necessarily schemes) (see [3] and [1]), in this note we restrict our considerations to schemes.

Remark 0.5. If $\mathcal{X}_1$ and $\mathcal{X}_2$ are two smooth proper models of X, then there is an obvious isomorphism $(\mathcal{X}_1)_K \cong (\mathcal{X}_2)_K$ and we also have an isomorphism $(\mathcal{X}_1)_k \cong (\mathcal{X}_2)_k$ ([1, Proposition 4.7]), but the models $\mathcal{X}_1$ and $\mathcal{X}_2$ themselves are not necessarily isomorphic (see [1, Section 4] for a further discussion).

Definition 0.6. A birational proper morphism $\mathcal{X} \to \mathcal{X}'$ between proper $\mathcal{O}_K$-schemes is a *simultaneous resolution* of $\mathcal{X}'$ if it induces the minimal resolution of a surface on each fiber over K and k. We use this notion only when all fibers of $\mathcal{X}$ are K3 surfaces.

Remark 0.7. Again, a simultaneous resolution is not unique in general.

0.4. **Corrected construction of ι in the relative setting.** Brief sketch: Although we do not know whether $\mathcal{X}$ is an ι-model, we can construct a birational proper morphism $\mathcal{X} \to \mathcal{X}''$ that is a simultaneous resolution of an ι-model $\mathcal{X}''$. Each connected component of the non-isomorphic locus of this morphism is not ι-stable. Using this property, we can modify $\mathcal{X} \to \mathcal{X}''$ into $\mathcal{X}^+ \to \mathcal{X}''$, where $\mathcal{X}^+$ is another smooth proper model to which ι extends.

Lemma 0.8. *Suppose $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ is as in Section 0.2. Let $\mathcal{D}$ be one of $\mathcal{D}_0$ and $\mathcal{D}_\infty$, and let $\mathcal{Z}$ be one of the sections (e.g. $\mathcal{W}_{00}$). Then, for non-negative integers m and n, the $\mathcal{O}_K$-module $M = H^0(\mathcal{X}, \mathcal{O}(m\mathcal{D} + n\mathcal{Z}))$ is free, and the natural morphisms*

$$M \otimes_{\mathcal{O}_K} k \to H^0(\mathcal{X}_K, \mathcal{O}(m\mathcal{D} + n\mathcal{Z})_K), \quad \textit{and}$$
$$M \otimes_{\mathcal{O}_K} k \to H^0(\mathcal{X}_k, \mathcal{O}(m\mathcal{D} + n\mathcal{Z})_k)$$

are isomorphisms.

Proof. Since $\dim_* H^0((\mathcal{X})_*, \mathcal{O}(m\mathcal{D} + n\mathcal{Z})_*)$ is independent of $* \in \{K, k\}$ (in fact, this value is the same for any elliptic fibration with section over a field), this follows from a well-known result on cohomology of fibers (see [2, Lemma 2.6(2)]). □

Let $\mathcal{X} \to \mathcal{X}'$ be the morphism to the Weierstrass model, constructed as the image of the projective morphism $\mathcal{X} \to \mathbb{P}^N_{\mathcal{O}_K}$ defined by using the linear system $H^0(\mathcal{X}, \mathcal{O}_\mathcal{X}(7\mathcal{D} + 3\mathcal{Z}))$. This formation commutes with taking the fibers over $\operatorname{Spec} \mathcal{O}_K$ by the lemma. Then $\mathcal{X}'$ is an ι-model. We can assume that $\mathcal{D}_0$ and $\mathcal{D}_\infty$ are the fibers over $0, \infty \in \mathbb{P}^1$ respectively.

Let $X \to Y_N \to Y_{N-1} \to \cdots \to Y_1 \to Y_0 = (\mathcal{X}')_K \xrightarrow{\Phi_K} \mathbb{P}^1_K$ be the sequence defined by the following.

- $Y_{i+1} \to Y_i$ is the blowup at the union $Z_i \subset Y_i$ of the singular points of Y_i lying above $\{0, \infty\} \subset \mathbb{P}^1_K$, when there exist such singular points. It follows that $X \to Y_i$ factors through Y_{i+1}.
- Y_N has no singular points lying above $\{0, \infty\}$.

Next we construct $\mathcal{Y}_N \to \mathcal{Y}_{N-1} \to \cdots \to \mathcal{Y}_1 \to \mathcal{Y}_0 = \mathcal{X}'$ by letting $\mathcal{Y}_{i+1}$ be the blowup at the closure $\mathcal{Z}_i$ of Z_i in $\mathcal{Y}_i$. It follows that each $\mathcal{Z}_i$ is a disjoint union of connected components each isomorphic to $\operatorname{Spec} \mathcal{O}_K$, and that the fibers of $(\mathcal{Y}_N)_k \to \mathbb{P}^1$ above 0 and ∞ are smooth (both of type IV*). We also note that $\mathcal{Y}_N$ is an ι-model, since the center of each blowup is ι-stable.

We show by induction on i ($1 \leq i \leq N$) that $\mathcal{X} \to \mathcal{Y}_{i-1}$ factors through $\mathcal{Y}_i$. By the universal property of the blowup, it suffices to show that the inverse image of $\mathcal{Z}_{i-1} \subset \mathcal{Y}_{i-1}$ in $\mathcal{X}$ is a Cartier divisor. We show that the inverse image is equal to $\mathcal{O}_\mathcal{X}(\sum_j a_j \mathcal{C}_j)$, where $\mathcal{C}_j$ are the components of $\mathcal{D}_0 \cup \mathcal{D}_\infty$ $(= \Phi^{-1}(\{0, \infty\}))$ and a_j are suitable non-negative integers. To show the equality, since $\mathcal{Z}_{i-1}$ and $\mathcal{O}_\mathcal{X}(\sum_j a_j \mathcal{C}_j)$ are flat over $\mathcal{O}_K$, it suffices to show it after passing to fibers over K and k. Then the claim follows from the fact that the description of the resolution of the RDP of type E_6 is independent of the base field (at least outside characteristic 2 and 3).

$$\begin{array}{ccccccc} \mathcal{X} & & & & & & \\ \downarrow & & & \searrow & & \searrow & \\ \mathcal{X}'' = \mathcal{Y}_N & \longrightarrow & \dots & \longrightarrow & \mathcal{Y}_1 & \longrightarrow & \mathcal{Y}_0 = \mathcal{X}' \end{array}$$

We thus obtain a morphism $\psi \colon \mathcal{X} \to \mathcal{X}'' := \mathcal{Y}_N$. This is a simultaneous resolution, and $\iota \circ \psi \colon \mathcal{X} \to \mathcal{X}''$ is another. We do not know whether $\mathcal{X}$ is an ι-model: this is equivalent to the existence of the dashed arrow in the following diagram.

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\psi} & \mathcal{X}'' \\ \downarrow{?} & & \downarrow{\iota} \\ \mathcal{X} & \xrightarrow{\psi} & \mathcal{X}'' \end{array}$$

The idea is to modify some part of $\mathcal{X}$ to obtain a new smooth proper model $\mathcal{X}^+$ for which the dashed arrow exists.

Let $\Phi_k^{-1}(t_i)$, $1 \leq i \leq M$, be the singular fibers of $\Phi_k \colon \mathcal{X}_k \to \mathbb{P}^1_k$ other than $(\mathcal{D}_0)_k$ and $(\mathcal{D}_\infty)_k$. By passing to an unramified extension of K, we can assume that all t_i are k-rational points of $\mathbb{P}^1_k$. Let $P_i \in \operatorname{Sing}((\mathcal{X}'')_k)$ be the corresponding singular points, which are also k-rational. We have the following.

- Each $Q \in \operatorname{Sing}((\mathcal{X}'')_K)$ is K-rational, and the intersection of $(\mathcal{X}'')_k$ with the closure of Q in $\mathcal{X}''$ is one of P_i.
- The set $\operatorname{Sing}((\mathcal{X}'')_K) \cup \operatorname{Sing}((\mathcal{X}'')_k)$ is closed and each of its connected components contains exactly one of P_i.

Therefore, a simultaneous resolution of $\mathcal{X}''$ is determined by its restrictions to neighborhoods of P_i.

Let $\sigma \colon \{1, \dots, M\} \to \{1, \dots, M\}$ be the involution characterized by $\iota(P_i) = P_{\sigma(i)}$. It is fixed-point-free since $t \mapsto -t$ is fixed-point-free outside $\{0, \infty\} \subset \mathbb{P}^1$.

For each M-tuple $\epsilon = (\epsilon_i) \in \{\pm 1\}^M$, let $\mathcal{X}^{(\epsilon)} \to \mathcal{X}''$ be the simultaneous resolution whose restriction to a neighborhood of P_i is isomorphic to $\psi \colon \mathcal{X} \to \mathcal{X}''$ if $\epsilon_i = 1$ and to $\iota \circ \psi \colon \mathcal{X} \to \mathcal{X}''$ if $\epsilon_i = -1$.

We observe that $\iota \colon \mathcal{X}''_K \to \mathcal{X}''_K$ extends to an isomorphism $\iota \colon \mathcal{X}^{(\epsilon)} \to \mathcal{X}^{(\iota^*(\epsilon))}$, where $\iota^*(\epsilon)$ is the M-tuple with i-th entry $-\epsilon_{\sigma(i)}$. In particular, $\mathcal{X}^\epsilon$ is an ι-model if and only if ϵ is ι-invariant, that is, $\iota^*(\epsilon) = \epsilon$, that is, $-\epsilon_{\sigma(i)} = \epsilon_i$. Since σ is fixed-point-free, there exists such ϵ.

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References

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